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Boundary value problem for abstract first order differential equation with integral condition

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Abstract

The present paper is devoted to the study of a boundary value problem for abstract first order linear differential equation with integral boundary conditions. We obtain necessary and sufficient conditions for the unique solvability and well-posedness. We also study the Fredholm solvability. Finally, we obtain a result of the stability of solution with respect to small perturbation.

Keywords: Abstract differential equation; C_0-semigroup; Fredholm property; Integral condition; Well-posed problem

1. Introduction

The first paper devoted to a nonlocal boundary value problem with integral conditions goes back to Cannon [2]. More general nonlocal conditions for different types of partial differential equations were considered later (see, e.g., [3–9,12,14–16]). The present paper is devoted to the study of a boundary value problem for abstract first order linear differential equation with integral boundary conditions. We obtain necessary and sufficient conditions for the unique solvability and well-posedness. We also study the Fredholm solvability. Finally, we consider the case when the differential equation is perturbed. The result obtained speaks of the stability of the solution with respect to small perturbation.

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The following notation is used in the paper: E is a Banach space with norm $\|.\|$; $\mathcal{L}(E)$ is the Banach algebra of bounded linear operators, mapping E into E, endowed with the usual operator norm; I is the identity operator in $\mathcal{L}(E)$; A is the generator of a semigroup $U(t)$ of class C_0, acting in the space E; $D(A)$ is the domain of definition of an operator A; $|||.|||$ is the graph norm on $D(A)$, i.e., $|||\Psi||| = \|\Psi\| + \|A\Psi\|$ for $\Psi \in D(A)$; $\mathbf{D}(A)$ is the Banach norm space $D(A)$ with the graph norm; $\rho(A)$, $\sigma(A)$ are the resolvent set and the spectrum on an operator A.

A function taking real or complex values, depending on whether the space E is real or complex, will be called a scalar function. A vector function is a function with values in some Banach space. By an operator function we mean a function with values in the Banach algebra $\mathcal{L}(E)$. All necessary information on vector integration can be found in [10]. Ordinarily we use the Riemann–Stieltjes integral; however, sometimes we have to resort to the Bochner integral. Integrals of operator-valued functions are considered a priori in the strong operator topology. The convergence of such an integral in the uniform operator topology (if it happens to be of interest) will be specifically stated.

The methods of the classical semigroup theory (see [1,10,11,13]) play an important role in the paper. The term “semigroup” will always mean a semigroup of bounded linear operators of class C_0.

2. Formulation of the problem

In Banach space E we consider the differential equation

$$\frac{du(t)}{dt} = Au(t), \quad 0 \leqslant t \leqslant T, \tag{1}$$

where A is a closed operator from E into E with dense domain $D(A)$ that generates a C_0-semigroup $U(t)$, $0 \leqslant t \leqslant T$.

Definition 1. The vector function $u(t) = U(t)f$, $0 \leqslant t \leqslant T$, corresponding to some element $f \in E$ is called a generalized solution of Eq. (1). If, in addition, $f \in D(A)$, then the solution $u(t) = U(t)f$ is said to be classical.

Remark 1. In the case, when $f \in D(A)$ obviously, f coincides with the initial state $u(0)$ of the corresponding solution $u(t)$.

Suppose that the initial state f is unknown, and consider the additional relation

$$\int_0^T w(t)u(t)\,dt = g \tag{2}$$

with a given element $g \in E$. Here $w(t)$ is a scalar measurable function of bounded variation on the segment $[0, T]$.

Remark 2. The integral occurring in (2) is well-defined in the sense of Bochner for any function $u(t) = U(t)f$.

Definition 2. A generalized solution of problem (1), (2) is defined to be a function $u(t) = U(t)f$, $0 \leqslant t \leqslant T$, corresponding to some element $f \in E$ and reducing relation (2) to a valid identity.

If, in addition, $f \in D(A)$, then the corresponding solution $u(t) = U(t)f$ of problem (1), (2) is called a classical solution.

3. Operator equation

From Definition 1, the solution of Eq. (1) is representable in the form $u(t) = U(t)f$.

Therefore, the function $u(t) = U(t)f$ satisfies condition (2) if and only if f satisfies the equation

$$\int_0^T w(t)U(t)f\,dt = g. \tag{3}$$

So, for $f \in E$, we have the operator equation $Bf = g$, where

$$Bf = \int_0^T w(t)U(t)f\,dt.$$

Lemma 1. *The operator B maps E into $D(A)$.*

Proof. If $f \in D(A)$, then $u(t) = U(t)f$ is a classical solution of Eq. (1) and $Af \in E$,

$$U(t)Af = AU(t)f = \frac{d}{dt}\big(U(t)f\big).$$

Therefore,

$$BAf = \int_0^T w(t)U(t)Af\,dt = \int_0^T w(t)AU(t)f\,dt.$$

Since A is closed operator, it follows that

$$\int_0^T w(t)AU(t)f\,dt = A\int_0^T w(t)U(t)f\,dt = ABf.$$

Hence $Bf \in D(A)$, and

$$ABf = \int_0^T w(t)\frac{d}{dt}\big(U(t)f\big)\,dt = \int_0^T w(t)\,d\big(U(t)f\big).$$

Using an integration by part in the sense of Stieltjes, we obtain

$$\int_0^T w(t)\,d\big(U(t)f\big) = w(T)U(T)f - w(0)f - \int_0^T U(t)f\,d\big(w(t)\big).$$

But if f is an arbitrary element from E, there exists a sequence $\{f_n\} \subset D(A)$ converging to f. Since B is a bounded operator, then $Bf_n \to Bf$ and

$$\begin{aligned}ABf_n &= w(T)U(T)f_n - w(0)f_n - \int_0^T U(t)f_n\,d\big(w(t)\big)\\ &\to w(T)U(T)f - w(0)f - \int_0^T U(t)f\,d\big(w(t)\big), \quad \text{as } n \to \infty.\end{aligned}$$

Since A is a closed operator, it follows that $Bf \in D(A)$ and

$$ABf = w(T)U(T)f - w(0)f - \int_0^T U(t)f\,d\big(w(t)\big). \tag{4}$$

So, for any $f \in E$ we conclude that $Bf \in D(A)$. □

Remark 3. If $g \in E \setminus D(A)$, then problem (1), (2) is unsolvable (in the sense of Definition 2).

Now both sides of Eq. (3) belong to $D(A)$, and consequently the operator $(\lambda I - A)$, where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), can be applied to them. Assume additionally that $\lambda \in \rho(A)$, this implies that $(\lambda I - A)^{-1}$ exists. Then from (4) we obtain the following equation for the element f:

$$Uf = Jf - Kf + \lambda Lf = \Psi. \tag{5}$$

Here J, K, L are bounded linear operators defined by the formulas

$$J = w(0)I, \tag{6}$$

$$K = w(T)U(T) + \int_0^T U(t)\,d\big(-w(t)\big), \tag{7}$$

$$L = \int_0^T w(t)U(t)\,dt. \tag{8}$$

(The integrals of operator-valued functions are defined in the strong topology of $\mathcal{L}(E)$.) The element $\Psi \in E$ in Eq. (5) has the form

$$\Psi = (\lambda I - A)g. \tag{9}$$

Theorem 1. *The function $u(t) = U(t)f$ is a solution of problem* (1), (2) *if and only if the element f satisfies Eq.* (5), *i.e., the equation $Uf = \Psi$, where $U = J - K + \lambda L$. Here J, K, L are the operators defined by the formulas* (6), (7), (8), *respectively, $\lambda \in \rho(A)$, and the element Ψ is defined by* (9). *In addition, problem* (1), (2) *is uniquely solvable for any g in $D(A)$ if and only if $U^{-1} \in \mathcal{L}(E)$.*

Remark 4. There is an obvious arbitrariness in the definition of the operator U and the element Ψ, connected with the choice of λ in $\rho(A)$. Since this choice does not play a crucial role in the reasoning below, we shall consider λ fixed henceforth, and the operator U and the element Ψ, thus, uniquely determined.

Proof. The equivalence of problem (1), (2) and Eq. (5) follows from the derivation of this equation. Let us show that for the unique solvability of problem (1), (2) it is necessary and sufficient that $U^{-1} \in \mathcal{L}(E)$, the sufficiency is obvious. Indeed, if $U^{-1} \in \mathcal{L}(E)$, then the function $u(t) = U(t)f$, where $f = U^{-1}\Psi$, and Ψ is defined by (9), will be the unique solution of problem (1), (2).

Necessity. Suppose that problem (1), (2) is uniquely solvable for any $g \in D(A)$. Then for any Ψ defined by (9) there exists a unique solution of (5). But if g runs through $D(A)$, then the element Ψ runs through the whole of E. Therefore, (5) has a unique solution f for any Ψ in E, i.e., the inverse operator $U^{-1} : E \to E$ exists, since the operator U is bounded, by Banach's theorem $U^{-1} \in \mathcal{L}(E)$, which was to be proved. □

4. Well-posedness of problem (1), (2)

The operator equation (5) is the main instrument in investigating the problem (1), (2). In particular, with its help the following theorem is obtained.

Theorem 2. *Suppose that problem* (1), (2) *is uniquely solvable for any $g \in D(A)$. Then its solution $u(t) = U(t)f$, corresponding to the element g satisfies the estimation*

$$\|u(t)\| \leqslant C(|||g|||) \tag{10}$$

with a constant $C > 0$ independent of g.

Proof. Since problem (1), (2) is uniquely solvable, by Theorem 1 its solution is representable in the form $u(t) = U(t)f$, where $f = U^{-1}\Psi$, $U^{-1} \in \mathcal{L}(E)$, and the element Ψ is given by formula (9). Therefore, $\|f\| \leqslant \|U^{-1}\|\|\Psi\|$. From this, taking the explicit form of Ψ and $u(t)$ into account, the estimate (10) follows immediately. □

We shall say that problem (1), (2) is well posed, if it has a unique solution $u(t) = U(t)f$, where $f \in E$ for any $g \in D(A)$. By Theorem 2, this solution satisfies the estimation (10) with a constant $C > 0$ independent of g.

Now we formulate the main definition.

Definition 3. Such a constant C will be called a well-posedness constant of problem (1), (2).

In other words, the problem of finding f (the vector function $u(t) = U(t)f$) for given g, well posed in the sense of Definition 3, is well posed in the sense of Hadamard on the pair of Banach spaces $(E, \mathbf{D}(A))$. The basic, although somewhat formal, well-posedness criterion is given by Theorem 1: problem (1), (2) is well posed if and only if $U^{-1} \in \mathcal{L}(E)$.

In the sequel we constantly use this fact, without special mention.

5. Fredholm property of problem (1), (2)

Let us consider how the uniqueness of solution and the well-posedness of problem (1), (2) are interrelated. We start with the following definition.

Definition 4. The kernel of problem (1), (2) is the set of element $u(t) = U(t)f$, $f \in E$, that are solutions of homogeneous condition (2) (i.e., when $g = 0$). If this set consists of the zero element alone, then the kernel of problem (1), (2) is said to be trivial.

It is easy to see that the kernel of problem (1), (2) coincides with the kernel of operator U defined in Theorem 1, and so it is a closed linear subspace of the space E. Further, if $u_1(t) = U(t)f_1$, $u_2(t) = U(t)f_2$ are solutions of problem (1), (2) corresponding to one and same g, then their difference $u(t) = u_1(t) - u_2(t)$ belongs to the kernel of this problem. Thus, for problem (1), (2) to be a uniquely solvable it is necessary and sufficient that its kernel be trivial; in particular, the kernel of a well-posed problem is, clearly, trivial. It turns out that the converse is also possible, when the triviality of the kernel implies the well-posedness of (1), (2).

Theorem 3. *Let the semigroup $U(t)$ be compact for $t > 0$, and let the function $w(t)$ be continuous on the right at $t = 0$ and $w(0) \neq 0$. Then*

1. *the kernel of problem* (1), (2) *is of finite dimension*;
2. *if the kernel of problem* (1), (2) *is trivial, then this problem is well posed.*

The proof of Theorem 3 is based on the following lemma.

Lemma 2. *Let the semigroup $U(t)$ be compact for $t > 0$, and let the function $w(t)$ be continuous on the right at $t = 0$ and $w(0) \neq 0$. Then the operators K and L defined by formulas* (6) *and* (7) *are compact.*

Proof. First, we show the compactness of L. Let $\varepsilon > 0$, and

$$L_\varepsilon = \int_\varepsilon^T w(t)U(t)\,dt.$$

Since $U(t)$ is compact operator for every $t > 0$, L_ε is compact. But the estimate

$$\begin{aligned}\|Lf - L_\varepsilon f\| &= \left\| \int_0^\varepsilon w(t)U(t)\,dt \right\| \\ &\leqslant \varepsilon\Big(\sup_{0\leqslant t\leqslant\varepsilon} \big\|U(t)f\big\| \sup_{0\leqslant t\leqslant T} \big|w(t)\big|\Big) \\ &\leqslant \varepsilon\Big(\sup_{0\leqslant t\leqslant\varepsilon} \big\|U(t)\big\| \sup_{0\leqslant t\leqslant T} \big|w(t)\big|\Big)\|f\|\end{aligned}$$

holds for any $f \in E$; moreover,

$$\varepsilon\Big(\sup_{0\leqslant t\leqslant\varepsilon} \big\|U(t)\big\| \sup_{0\leqslant t\leqslant T} \big|w(t)\big|\Big)\|f\| \to 0 \quad \text{as } \varepsilon \to 0^+,$$

and therefore L is compact as a uniform limit of compacts operators. The compactness of the operator

$$K = w(T)U(T) + \int_0^T U(t)\,d\big(-w(t)\big)$$

is proved analogously.

The lemma is proved. □

Proof of Theorem 3. Let us associate the operator $U = J - K + \lambda L$ with problem (1), (2). Since $w(0) \neq 0$, it follows that $J^{-1} \in \mathcal{L}(E)$ and the operators K and L are compact, the well-known Fredholm theorems are valid for U. In particular, the operator U has a finite-dimensional kernel, the triviality of which is equivalent to $U^{-1} \in \mathcal{L}(E)$. But this is exactly the claim of our theorem, since the kernel of U coincides with the kernel of problem (1), (2). □

Remark 5. In essence, in Theorem 3 it is shown that the operator U corresponding to problem (1), (2) is Fredholm of index zero. In such situations it is customary to say that the problem (1), (2) itself is Fredholm of index zero.

6. Well-posedness conditions

In this section we give examples of conditions guaranteeing the well-posedness of problem (1), (2).

Theorem 4. *Let $w(t)$ be a nonnegative nonincreasing function for $t \in [0, T]$ such that $w(t) > 0$ as $t \to 0^+$, and let the semigroup $U(t)$ generated by the operator A satisfy the estimate $\|U(t)\| \leqslant M \exp(-\beta t)$ with constants $M \geqslant 1$, $\beta > 0$. Then problem* (1), (2) *is well posed* (*in the sense of Definition* 3).

Proof. Equation (5) becomes $(w(0)I - K)f = \Psi$, where under the assumptions of the theorem, the term λL in (5) may be dropped, since the spectrum $\sigma(A)$ is contained in the half-plane $\{\lambda\colon \operatorname{Re}\lambda \leqslant -\beta < 0\}$. It suffices to show that $\|K\|_1 < w(0)$ in some norm $\|.\|_1$ equivalent to $\|.\|$. Indeed, then $(w(0)I - K)^{-1} \in \mathcal{L}(E)$, and by Theorem 1, problem (1), (2) is well posed. We defined the equivalent norm $\|.\|_1$ in the space E by the formula $\|h\|_1 = \sup_{t \geqslant 0} \|U(t)\exp(\beta t)\|$ $(h \in E)$. It is known that then $\|U(t)\|_1 \leqslant \exp(-\beta t)$ (see [10]).

Thus, for this equivalent norm, we have

$$\begin{aligned}
\|K\|_1 &= \left\| w(T)U(T) + \int_0^T U(t)\, d\bigl(-w(t)\bigr) \right\|_1 \\
&\leqslant \bigl\| w(T)U(T) \bigr\|_1 + \left\| \int_0^T U(t)\, d\bigl(-w(t)\bigr) \right\|_1 \\
&\leqslant w(T)\exp(-\beta T) + \int_0^T \exp(-\beta t)\, d\bigl(-w(t)\bigr).
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
\|K\|_1 &\leqslant w(T)\exp(-\beta T) - w(T)\exp(-\beta T) + w(0) - \beta \int_0^T \exp(-\beta t) w(t)\, dt \\
&= w(0) - \beta \int_0^T \exp(-\beta t) w(t)\, dt.
\end{aligned}$$

Since $\beta\int_0^T \exp(-\beta t)w(t)\,dt$ is positive under the assumptions of theorem, it follows that

$$w(0)-\beta\int_0^T \exp(-\beta t)w(t)\,dt < w(0).$$

Consequently problem (1), (2) is well posed, and the theorem is proved. □

Remark 6. In the case when $T=+\infty$ this result coincides with [17].

Theorem 5. *Let $w(t)$ be a nonnegative nonincreasing function for $t\in[0,T]$ such that $w(t)>0$ as $t\to 0^+$, and let the semigroup $U(t)$ generated by the operator A be positive and compact for $t>0$. Assume that the spectrum of A lies in the half-plane $\{\lambda\in\mathbb{C}\colon \operatorname{Re}\lambda<0\}$. Then problem* (1)*,* (2) *is well posed* (*in the sense of Definition* 3)*.*

Proof. Since the semigroup $U(t)$ is positive, the requirement on the spectrum $\sigma(A)$ means that $s(A)<0$, where $s(A)\equiv\sup\{\operatorname{Re}\lambda\colon \lambda\in\sigma(A)\}$ is the spectral boundary of A. This implies that, in particular, that the resolvent $(\lambda I-A)^{-1}$ exists for $\lambda=0$ and is positive operator. Then the term λL in (5) may be dropped. Hence Eq. (5) becomes $U=J-K$, where

$$J=w(0)I,$$
$$K=w(T)U(T)+\int_0^T U(t)\,d\big(-w(t)\big).$$

Furthermore, by the compactness of $U(t)$, the requirement $\|U(t)\|\leqslant M\exp(-\beta t)$ is equivalent to $s(A)<0$. Using Theorem 4, we obtain the proof. Consequently problem (1), (2) is well posed, and the theorem is proved. □

7. Perturbed problem

The remaining part of the paper will be dedicated to the question of small perturbations of problem (1), (2). We consider the case when Eq. (1) is perturbed. Thus, suppose that along with Eq. (1) there is the equation

$$\frac{du(t)}{dt}=(A+B)u(t),\quad 0\leqslant t\leqslant T, \tag{11}$$

in this case, we have a problem (11), (2) instead of problem (1), (2). Here $B\in\mathcal{L}(E)$. We shall also write $\tilde{A}=A+B$. Since B is a bounded linear operator, the operator $\tilde{A}$ with $D(\tilde{A})=D(A)$ is the generator of some C_0-semigroup $\tilde{U}(t)$ [13].

Theorem 6. *Let the problem* (1), (2) *be well posed. Then there exists a number $\delta>0$, depend only on A and $w(t)$, such that if $\|B\|\leqslant\delta$ for $0\leqslant t\leqslant T$, then problem* (11), (2) *is also well posed. For any $\varepsilon>0$ the indicated δ can be chosen such that*

$$\big\|\tilde{u}(t)-u(t)\big\|\leqslant\varepsilon\big(|||g|||\big), \tag{12}$$

where $\tilde{u}(t)=\tilde{U}(t)\tilde{f}$, $u(t)=U(t)f$ are solutions of problems (1), (2); (11), (2), *respectively, corresponding to given $g\in D(A)$.*

Remark 7. The estimate (12) means that $\|\tilde{u}(t) - u(t)\| \to 0$ as $\|B\| \to 0$. Thus, the solutions of well-posed problem (1), (2) are stable under small perturbations of (1).

Proof of Theorem 6. Without loss of generality we may suppose that $\delta \leqslant 1$. We choose constants M and α ($M \geqslant 1$, $\alpha \in \mathbb{R}$) such that the estimate $\|U(t)\| \leqslant M\exp(\alpha t)$ holds for the semigroup $U(t)$. In this case the following relations hold for the perturbed semigroup $\tilde{U}(t)$ [13]:

$$\|\tilde{U}(t)\| \leqslant M\exp\big((\alpha + M\|B\|)t\big),$$
$$\|\tilde{U}(t) - U(t)\| \leqslant M\exp(\alpha t)\big[\exp\big(M\|B\|t\big) - 1\big].$$

Since $\|B\| \leqslant \delta \leqslant 1$, for $0 \leqslant t \leqslant T$ we have

$$\|\tilde{U}(t)\| \leqslant M_1, \qquad \|\tilde{U}(t) - U(t)\| \leqslant M_2\big(\exp(MT\delta) - 1\big). \tag{13}$$

(The symbols $M_1, M_2, \ldots$ denote constants dependent only on the data of problem (1), (2), i.e., on A, $w(t)$.)

We consider Eq. (5) for problems (1), (2) and (11), (2). Let $U = J - K + \lambda L$, and $\tilde{U} = \tilde{J} - \tilde{K} + \lambda\tilde{L}$ be the corresponding operators. (The operators $\tilde{J}$, $\tilde{K}$, $\tilde{L}$ are defined by the formulas (6), (7), (8) with the replacement of $U(t)$ by $\tilde{U}(t)$.) The number λ in the operators U and $\tilde{U}$ is chosen the same. Since $s(A) \leqslant \alpha$ and $s(A+B) \leqslant \alpha + M\|B\| \leqslant \alpha + M$, we may assume that, for example, $\lambda = \alpha + M + 1$. Using the explicit form of the operators U and $\tilde{U}$, the estimate (13), it is not difficult to show that

$$\|\tilde{U} - U\| \leqslant M_3\big(\big(\exp(MT\delta) - 1\big)\big).$$

Consequently, $\|\tilde{U} - U\|$ can be made as small as desired, provided that δ is sufficiently small. By the hypothesis of the theorem, problem (1), (2) is well posed, and thus, $U^{-1} \in \mathcal{L}(E)$. Choosing δ such that $\|\tilde{U} - U\| \leqslant \|U^{-1}\|^{-1}$, we get that $\tilde{U}^{-1} \in \mathcal{L}(E)$. But this is equivalent to well-posedness of problem (11), (2).

To prove the estimate (12), notice that $f = U^{-1}\Psi$, $\tilde{f} = \tilde{U}^{-1}\tilde{\Psi}$, where Ψ and $\tilde{\Psi}$ are defined by formula (9). Using the explicit form of $\tilde{\Psi}$, Ψ and relation $\|B\| \leqslant \delta$, we get

$$\|\tilde{\Psi} - \Psi\| \leqslant \delta\big(\|g\|\big),$$

and, consequently

$$\|\tilde{\Psi}\| \leqslant \|\Psi\| + \|\tilde{\Psi} - \Psi\| \leqslant M_4\big(|||g|||\big).$$

But then

$$\begin{aligned}\|\tilde{f} - f\| &\leqslant \|\tilde{U}^{-1} - U^{-1}\|\|\tilde{\Psi}\| + \|U^{-1}\|\|\tilde{\Psi} - \Psi\| \\ &\leqslant M_4\|\tilde{U}^{-1} - U^{-1}\||||g||| + \delta\|U^{-1}\|\|g\| \\ &\leqslant M_5\big[\|\tilde{U}^{-1} - U^{-1}\| + \delta\big]\big(|||g|||\big).\end{aligned}$$

Since

$$\|\tilde{U} - U\| \to 0 \quad \text{as } \delta \to 0$$

and the operation of passage to the inverse operator is continuous in the norm of $\mathcal{L}(E)$,

$$\|\tilde{U}^{-1} - U^{-1}\| \to 0 \quad \text{as } \delta \to 0.$$

Now the estimate (12) is obvious. Indeed,

$$\begin{aligned}\left\|\tilde{u}(t)-u(t)\right\| &= \left\|\tilde{U}(t)\tilde{f}-U(t)f+\tilde{U}(t)f-\tilde{U}(t)f\right\| \\ &= \left\|\tilde{U}(t)(\tilde{f}-f)+\big(\tilde{U}(t)-U(t)\big)f\right\| \\ &\leqslant \left\|\tilde{U}(t)\right\|\left\|(\tilde{f}-f)\right\|+\left\|\tilde{U}(t)-U(t)\right\|\|f\| \\ &\leqslant M_6\big[\left\|\tilde{U}^{-1}-U^{-1}\right\|+\delta\big]\big(\|\!|g|\!\|\big)+CM_2\big(\exp(MT\delta)-1\big)\big(\|\!|g|\!\|\big).\end{aligned}$$

The theorem is proved. □

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